A focusing and defocusing complex short pulse equation

Baofeng Feng

Department of Mathematics,
The University of Texas Rio Grande Valley

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Outline of the talk

- Derivation of the complex short pulse (CSP) equation from nonlinear optics
- Bright, breather and rogue wave solutions to the focusing CSP equation.
- Dark soliton solution to the defocusing CSP equation
- Semi- and fully discrete analogues of the CSP equation
- Conclusion and further topics

Joint work with:

K. Maruno (Waseda University), Y. Ohta (Kobe University),

L. Ling (South China Univ. of Tech.), Z. Zhu (Shanghai Jiaotong Univ.)

Review on the Nonlinear Schrödinger equation

Nonlinear Schrödinger (NLS) equation

$$\mathbf{i}q_t + q_{xx} + \sigma 2|q|^2 q = 0\,, \quad \sigma = \pm 1$$

- $\sigma = 1$: focusing case, possessing bright soliton
- $\sigma = -1$: defocusing case, possessing dark soliton
- Rogue wave solution for the focusing NLS equation

Nonlinear Schrödinger (NLS) equation

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Integrable semi-discrete NLS equation: Ablowitz-Ladik (AL) lattice

$$\mathrm{i}\frac{\partial q_n}{\partial t} + (1 + \sigma|q_n|^2)(q_{n-1} + q_{n+1}) = 0\,, \quad \sigma = \pm 1$$

Coupled Nonlinear Schrödinger (CNLS) equation

$$\mathrm{i}q_{1,t} + q_{1,xx} + 2(|q_1|^2 + B|q_2|^2)q_1 = 0\,,$$

$$\mathrm{i}q_{2,t} + q_{2,xx} + 2(|q_2|^2 + B|q_1|^2)q_2 = 0\,.$$

- The parameter B is related to the ellipticity angle θ as

$$B = \frac{2 + 2\sin^2\theta}{2 + \cos^2\theta}\,.$$

- For a linearly birefringent fiber ($\theta = 0$), $B = \frac{2}{3}$, for a circularly birefringent fiber ($\theta = \pi/2$), $B = 2$. Only when $B = 1$, it is integrable (Manakov system)

Short pulse equation

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}$$

- Schäfer & Wayne(2004): Derived from Maxwell equation on the setting of ultra-short optical pulse in silica optical fibers.
- Sakovich & Sakovich (2005): A Lax pair of WKI type, linked to sine-Gordon equation through hodograph transformation;
- Matsuno (2007): Multisoliton solutions through Hirota's bilinear method
- FMO (2010): Integrable semi- and fully discretizations.

Complex short pulse equation

$$q_{xt} + q + \frac{1}{2}\sigma(|q|^2 q_x)_x = 0, \quad (\sigma = \pm 1)$$

- It is integrablewhich can be viewed as an analogue of the NLS equation in the ultra-short pulse region.
- It is more natural and appropriate in describing the propagation of the ultra-short pulses in compared with the short pulse equation
- $\sigma = 1$: focusing case, bright soliton, breather and rogue wave solutions
- $\sigma = -1$: defocusing case, dark soliton

Coupled complex short pulse equation

$$q_{1,xt} + q_1 + \frac{1}{2}\left((|q_1|^2 + B|q_2|^2)q_{1,x}\right)_x = 0\,,$$

$$q_{2,xt} + q_2 + \frac{1}{2}\left((|q_2|^2 + B|q_1|^2)q_{2,x}\right)_x = 0\,.$$

- The parameter B is related to the ellipticity angle θ same as the NLS equation.
- For a linearly birefringent fiber ($\theta = 0$), $B = \frac{2}{3}$, for a circularly birefringent fiber ($\theta = \pi/2$), $B = 2$.
- Similar to the Manakov system, only when $B = 1$, it is integrable.

Maxwell's Equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,, \quad \nabla \times \mathbf{H} = -\frac{\partial \mathbf{D}}{\partial t}\,.$$
$$\mathbf{D} = \epsilon \mathbf{E}\,, \quad \mathbf{B} = \mu \mathbf{H}\,, \quad \mathbf{D} = \mathbf{E} + \mathbf{P}\,.$$

ϵ: permittivity, μ: permeability. In vacuum, $c^2 = 1/(\epsilon_0 \mu_0)$.

Maxwell's Equations:

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ϵ: permittivity, μ: permeability. In vacuum, $c^2 = 1/(\epsilon_0 \mu_0)$.
In the frequency-dependent media,

$$\mathbf{D} = \epsilon \star \mathbf{E}\,, \quad \mathbf{B} = \mu \star \mathbf{H}\,.$$

where $\epsilon = \epsilon_0(1 + \chi^{(1)}(t))$. In the frequency domain

$$\tilde{\mathbf{D}} = \tilde{\epsilon}(\omega)\tilde{\mathbf{E}}\,, \quad \tilde{\mathbf{B}} = \tilde{\mu}(\omega)\tilde{\mathbf{H}}\,.$$

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\mathbf{E}_{tt} = \mu_0 \mathbf{P}_{tt}\,,$$

The induced polarization $\mathbf{P}(\mathbf{r}, t) = \mathbf{P}_L(\mathbf{r}, t) + \mathbf{P}_{NL}(\mathbf{r}, t)$.

Derivation of complex short pulse equation (II)

Assuming

$$\mathbf{E} = \frac{1}{2}\mathrm{e}_1 \left(E(z,t) + c.c. \right) ,$$

where $E(z,t)$ is a complex-valued function.

$$\tilde{E}_{zz}(z,\omega) + \tilde{\epsilon}(\omega)\frac{\omega^2}{c^2}\tilde{E}(z,\omega) = 0 \,,$$

where $\tilde{E}(z,\omega)$ is the Fourier transform of $E(z,t)$ defined as

$$\tilde{E}(z,\omega) = \int_{-\infty}^{\infty} E(z,t)e^{\mathrm{i}\omega t}\, dt \,,$$

$$\tilde{\epsilon}(\omega) = 1 + \tilde{\chi}^{(1)}(\omega) \,.$$

where $\tilde{\chi}^{(1)}(\omega)$ is the Fourier transform of $\chi^{(1)}(t)$

$$\tilde{\chi}^{(1)}(\omega) = \int_{-\infty}^{\infty} \chi^{(1)}(t)e^{\mathrm{i}\omega t}\, dt \,.$$

In the range ultra-short pulse, we approximate the response function $\chi^{(1)}(\lambda)$ by

$$\tilde{\chi}^{(1)}(\lambda) = \tilde{\chi}_0^{(1)} \mp \tilde{\chi}_2^{(1)} \lambda^2 \,, \quad \tilde{\chi}_2^{(1)} > 0, \lambda = \frac{2\pi c}{\omega} \,.$$

For Kerr media with cubic nonlinearity, $P_{NL}(z,t) = \epsilon_0 \epsilon_{NL} E(z,t)$
$\epsilon_{NL} = \frac{3}{4}\chi^{(3)}_{xxxx}|E(z,t)|^2$.

$$\tilde{E}_{zz} + \frac{1+\tilde{\chi}_0^{(1)}}{c^2}\omega^2 \tilde{E} \mp (2\pi)^2 \tilde{\chi}_2^{(1)} \tilde{E} + \epsilon_{NL}\frac{\omega^2}{c^2}\tilde{E} = 0 \,.$$

Derivation of complex short pulse equation (V)

Applying the inverse Fourier transform yields a single nonlinear wave equation

$$E_{zz} - \frac{1}{c_1^2} E_{tt} = \pm \frac{1}{c_2^2} E + \frac{3}{4} \chi^{(3)}_{xxxx} \left(|E|^2 E\right)_{tt} = 0 \,.$$

Applying multiple scale expansion,

$$E(z,t) = \epsilon E_0(\phi, z_1, z_2, \cdots) + \epsilon^2 E_1(\phi, z_1, z_2, \cdots) + \cdots \,,$$

where ϵ is a small parameter, ϕ and z_n are the scaled variables defined by

$$\phi = \frac{t - \frac{x}{c_1}}{\epsilon}, \quad z_n = \epsilon^n z \,.$$

$$-\frac{2}{c_1} \frac{\partial^2 E_0}{\partial \phi \partial z_1} = \pm \frac{1}{c_2^2} E_0 + \frac{3}{4} \chi^{(3)}_{xxxx} \frac{\partial}{\partial \phi} \left(|E_0|^2 \frac{\partial E_0}{\partial \phi} \right) \,.$$

Focusing and defocusing compex short pulse equation

By a scale transformation

$$x = \frac{c_1}{2}\phi, \quad t = c_2 z_1, \quad q = \frac{c_1\sqrt{6c_2\chi^{(3)}_{xxxx}}}{4}E_0$$

we have

$$q_{xt} \pm q + \frac{1}{2}\left(|q|^2 q_x\right)_x = 0$$

$$q_{xt} + q + \frac{1}{2}\sigma\left(|q|^2 q_x\right)_x = 0\,, \quad \sigma = \pm 1.$$

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$$q_{xt} \pm q + \frac{1}{2}\left(|q|^2 q_x\right)_x = 0$$

$$q_{xt} + q + \frac{1}{2}\sigma\left(|q|^2 q_x\right)_x = 0\,, \quad \sigma = \pm 1.$$

Coupled complex short pulse equation of mixed type

$$\begin{cases} q_{1,xt} + q_1 + \frac{1}{2}\left((\sigma_1|q_1|^2 + \sigma_2|q_2|^2)q_{1,x}\right)_x = 0, \\ q_{2,xt} + q_2 + \frac{1}{2}\left((\sigma_1|q_1|^2 + \sigma_2|q_2|^2)q_{2,x}\right)_x = 0 \end{cases}$$

- focusing-focusing ($\sigma_1 = \sigma_2 = 1$); defocusing-defocusing ($\sigma_1 = \sigma_2 = -1$) and focusing-defocusing ($\sigma_1 = 1; \sigma_2 = -1$).
- Bright, dark and bright-dark soliton solutions and rogue wave solution

Complex coupled dispersionless (CCD) equation

$$\begin{cases} q_{ys} = \rho q, \\ \rho_s \pm \frac{1}{2}(|q|^2)_y = 0 \end{cases}$$

- Konno K, Kakuhata H. J Phys Soc Jpn 1995, 64, 2707, 1996;65:713
- K. Konno, Appl. Anal., 57, 209 (1995).
- Only the positive sign was studied

From the complex coupled dispersioless equation to the complex short pulse equation

$$\begin{cases} q_{ys} = \rho q, \\ \rho_s \pm \frac{1}{2}(|q|^2)_y = 0 \end{cases}$$

We define a hodograph transformation

$$dx = \rho dy \mp \frac{1}{2}|q|^2 d\, s, \quad dt = -d\, s,$$

then we have

$$\partial_y = \rho^{-1}\partial_x, \quad \partial_s = -\partial_t \mp \frac{1}{2}|q|^2\partial_x$$

Accordingly, the equation $q_{ys} = \rho q$ gives the

$$\partial_x(-\partial_t \mp \frac{1}{2}|q|^2\partial_x)q = q,$$

$$q_{xt} + q \pm \frac{1}{2}(|q|^2 q_x)_x = 0.$$

Bilinear equations of the focusing complex short pulse equation

Theorem

The focusing complex short pulse equation

$$q_{xt} + q + \frac{1}{2}\left(|q|^2 q_x\right)_x = 0$$

can be derived from bilinear equations

$$D_s D_y f \cdot g = fg\,, \quad D_s^2 f \cdot f = \frac{1}{2}|g|^2\,,$$

through the hodograph transformation

$$x = y - 2(\ln f)_s\,, \quad t = -s$$

and the dependent variable transformation $q = \frac{g}{f}$

Multi bright soliton solution to the focusing complex short pulse equation

Theorem

The CSP equation admits multi-soliton solution

$$f = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}_{2N \times 2N}, \quad g = \begin{vmatrix} A & I & \Phi^T \\ -I & B & 0^T \\ 0 & C_1 & 0 \end{vmatrix}_{(2N+1)\times(2N+1)},$$

where the elements defined respectively by

$$a_{ij} = \frac{1}{2(p_i^{-1} + {p^*}_j^{-1})} e^{\xi_i + {\xi^*}_j}, \quad b_{ij} = \frac{\alpha_i \alpha_j^*}{2(p_j^{-1} + {p^*}_i^{-1})}$$

$$\xi_i = p_i y + \frac{1}{p_i} s + \xi_{i0}, \; \xi_j^* = p_j^* y + \frac{1}{p_j^*} s + \xi_{j0}^*,$$

One-soliton to the focusing complex SP equation

$$f = 1 + \frac{1}{4}\frac{|\alpha_1|^2 (p_1 \bar{p}_1)^2}{(p_1 + \bar{p}_1)^2} e^{\eta_1 + \bar{\eta}_1}\,, \quad g = \alpha_1 e^{\eta_1}\,.$$

Let $p_1 = p_{1R} + \mathrm{i} p_{1I}$

$$q = \frac{\alpha_1}{|\alpha_1|}\frac{2p_{1R}}{|p_1|^2} e^{\mathrm{i}\eta_{1I}} \,\mathrm{sech}\left(\eta_{1R} + \eta_{10}\right)\,,$$

$$x = y - \frac{2p_{1R}}{|p_1|^2}\left(\tanh\left(\eta_{1R} + \eta_{10}\right) + 1\right)\,, \quad t = -s\,,$$

When $p_{1R} < p_{1I}$, the solution is a smooth envelop soliton; when $p_{1R} = p_{1I}$, the solution becomes a cuspon solition.

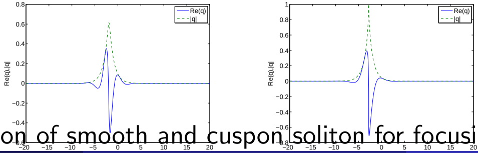


Figure: Illustration of smooth and cuspon soliton for focusing CSP equation

Define the following tau-functions for two-component KP hierarchy,

$$f_{mn} = \begin{vmatrix} A & I \\ -I & B \end{vmatrix},$$

$$g_{mn} = \begin{vmatrix} A & I & \Phi^T \\ -I & B & 0^T \\ 0 & -\bar{\Psi} & 0 \end{vmatrix}, \quad h_{mn} = \begin{vmatrix} A & I & 0^T \\ -I & B & \Psi^T \\ -\bar{\Phi} & 0 & 0 \end{vmatrix},$$

where A and B are $N \times N$ matrices whose elements are

$$a_{ij} = \frac{1}{p_i + \bar{p}_j} \left(-\frac{p_i}{\bar{p}_j} \right)^n e^{\xi_i + \bar{\xi}_j}, \quad b_{ij} = \frac{1}{q_i + \bar{q}_j} \left(-\frac{q_i}{\bar{q}_j} \right)^m e^{\eta_i + \bar{\eta}_j},$$

with

$$\xi_i = \frac{1}{p_i} x_{-1} + p_i x_1 + \xi_{i0}, \quad \bar{\xi}_j = \frac{1}{\bar{p}_j} x_{-1} + \bar{p}_j x_1 + \bar{\xi}_{j0},$$

$$\eta_i = q_i y_1 + \eta_{i0}, \quad \bar{\eta}_j = \bar{q}_j y_1 + \bar{\eta}_{j0},$$

Two-component KP hierarchy and its Gram-type solution

Φ, Ψ, $\bar{\Phi}$ and $\bar{\Psi}$ are N-component row vectors

$$\Phi = \left(p_1^n e^{\xi_1}, \cdots, p_N^n e^{\xi_N}\right), \; \bar{\Phi} = \left((-\bar{p}_1)^{-n} e^{\bar{\xi}_1}, \cdots, (-\bar{p}_N)^{-n} e^{\bar{\xi}_N}\right),$$

$$\Psi = \left(q_1^m e^{\eta_1}, \cdots, q_N^m e^{\eta_N}\right), \bar{\Psi} = \left((-\bar{q}_1)^{-m} e^{\bar{\eta}_1}, \cdots, (-\bar{q}_N)^{-m} e^{\bar{\eta}_N}\right).$$

Then the following bilinear equations hold

$$\frac{1}{2} D_{x_1} D_{y_1} f_{nm} \cdot f_{nm} = -g_{nm} h_{nm},$$

$$D_{x_{-1}} g_{nm} \cdot f_{nm} = g_{n-1,m} f_{n+1,m},$$

$$(D_{x_1} D_{x_{-1}} - 2) g_{nm} \cdot f_{nm} = -D_{x_1} g_{n-1,m} \cdot f_{n+1,m},$$

$$D_{x_1} g_{n,m+1} \cdot f_{n+1,m} = g_{n+1,m+1} f_{nm}.$$

Reductions to the CSP equation (I)

Recall the bilinear equation of the CSP equation

$$D_sD_yf\cdot g=fg\,,\quad D_s^2f\cdot f=\frac{1}{2}|g|^2\,,$$

Task: How to get them from the following bilinear equations of two-component KP?

$$\frac{1}{2}D_{x_1}D_{y_1}f_{nm}\cdot f_{nm}=-g_{nm}h_{nm}\,,$$

$$D_{x_{-1}}g_{nm}\cdot f_{nm}=g_{n-1,m}f_{n+1,m}\,,$$

$$(D_{x_1}D_{x_{-1}}-2)g_{nm}\cdot f_{nm}=-D_{x_1}g_{n-1,m}\cdot f_{n+1,m}\,,$$

$$D_{x_1}g_{n,m+1}\cdot f_{n+1,m}=g_{n+1,m+1}f_{nm}\,.$$

Reductions to the CSP equation (II)

Under the condition $q_j = \bar{p}_j, \quad \bar{q}_j = p_j$ we have

$$f_{n+1,m+1} = f_{nm}, \quad g_{n+1,m+1} = -g_{nm}\,,$$

$$\partial_{x_1} f_{nm} = \partial_{y_1} f_{nm}, \quad \partial_{x_1} g_{nm} = \partial_{y_1} g_{nm}\,.$$

it then follows

$$\begin{aligned}(D_{x_1}D_{x_{-1}} - 2)g_{nm}\cdot f_{nm} &= D_{x_1}g_{n,m+1}\cdot f_{n+1,m}\\ &= g_{n+1,m+1}f_{nm}\\ &= -g_{nm}f_{nm}\end{aligned}$$

from

$$(D_{x_1}D_{x_{-1}} - 2)g_{nm}\cdot f_{nm} = -D_{x_1}g_{n-1,m}\cdot f_{n+1,m}\,,$$

$$D_{x_1}g_{n,m+1}\cdot f_{n+1,m} = g_{n+1,m+1}f_{nm}\,.$$

Reductions to the CSP equation (III)

$$\partial_{x_1} f_{nm} = \partial_{y_1} f_{nm}, \quad \partial_{x_1} g_{nm} = \partial_{y_1} g_{nm} \,.$$

From

$$\frac{1}{2} D_{x_1} D_{y_1} f_{nm} \cdot f_{nm} = -g_{nm} h_{nm} \,,$$

it then follows

$$\frac{1}{2} D_{x_1}^2 f_{nm} \cdot f_{nm} = -g_{nm} h_{nm} \,.$$

Let $f = f_{00}$, $g = g_{00}$, $h = h_{00}$, the above bilinear equations read

$$(D_{x_1} D_{x_{-1}} - 1) g \cdot f = 0 \,,$$

$$\frac{1}{2} D_{x_1}^2 f \cdot f = -gh \,.$$

Reductions to the CSP equation (IV)

By taking

$$\bar{p}_j = p_j^*, \quad \bar{\xi}_{j0} = \xi_{j0}^*, \quad \bar{\eta}_{j0} = \eta_{j0}^*,$$

we can easily check that f is real and $h = -g^*$. Then

$$(D_{x_1} D_{x_{-1}} - 1) g \cdot f = 0,$$

$$D_{x_1}^2 f \cdot f = 2|g|^2.$$

Reductions to the CSP equation (IV)

By taking

$$\bar{p}_j = p_j^*, \quad \bar{\xi}_{j0} = \xi_{j0}^*, \quad \bar{\eta}_{j0} = \eta_{j0}^*,$$

we can easily check that f is real and $h = -g^*$. Then

$$(D_{x_1} D_{x_{-1}} - 1) g \cdot f = 0,$$

$$D_{x_1}^2 f \cdot f = 2|g|^2.$$

By variable transformation

$$s = 2(x_1 + y_1), \quad y = \frac{1}{2}(x_{-1} + y_{-1}),$$

we arrive at the bilinear equations for the CSP equation. The multi-soliton solution can be obtained by a reparametrization

$$p_i \to 2p_i^{-1}, \quad p_i^* \to 2p_i^{*-1},$$

Lax pair for the CCD and CSP equations

It is known that the CCD equation admits the following Lax pair

$$\Psi_y = U(\rho, q; \lambda)\Psi, \quad \Psi_s = V(q; \lambda)\Psi,$$

where

$$U(\rho, q; \lambda) = \begin{bmatrix} -\frac{\mathrm{i}\rho}{\lambda} & -\frac{q_y^*}{\lambda} \\ \frac{q_y}{\lambda} & \frac{\mathrm{i}\rho}{\lambda} \end{bmatrix}, \; V(q; \lambda) = \begin{bmatrix} \frac{\mathrm{i}}{4}\lambda & \frac{\mathrm{i}q^*}{2} \\ \frac{\mathrm{i}q}{2} & -\frac{\mathrm{i}}{4}\lambda \end{bmatrix}$$

Through the reciprocal transformation:

$$\mathrm{d}x = \rho \mathrm{d}y - \frac{1}{2}|q|^2 \mathrm{d}s, \quad \mathrm{d}t = -\mathrm{d}s,$$

one can obtain the CSP equation and its Lax pair:

$$\Psi_x = \begin{bmatrix} -\frac{\mathrm{i}}{\lambda} & -\frac{q_x^*}{\lambda} \\ \frac{q_x}{\lambda} & \frac{\mathrm{i}}{\lambda} \end{bmatrix} \Psi,$$

$$\Psi_t = \begin{bmatrix} -\frac{\mathrm{i}}{4}\lambda + \frac{\mathrm{i}|q|^2}{2\lambda} & -\frac{\mathrm{i}q^*}{2} + \frac{|q|^2 q_x^*}{2\lambda} \\ -\frac{\mathrm{i}q}{2} - \frac{|q|^2 q_x}{2\lambda} & \frac{\mathrm{i}}{4}\lambda - \frac{\mathrm{i}|q|^2}{2\lambda} \end{bmatrix} \Psi.$$

Darboux transformation for the focusing CSP equation

Theorem

The Darboux matrix

$$T = I + \frac{\lambda_1^* - \lambda_1}{\lambda - \lambda_1^*} P_1, \;\; P_1 = \frac{|y_1\rangle\langle y_1|}{\langle y_1|y_1\rangle}, \;\; |y_1\rangle = \begin{bmatrix} \psi_1(x,t;\lambda_1) \\ \phi_1(x,t;\lambda_1) \end{bmatrix}$$

can convert the Lax pair of the CSP eq. $\Psi_y = U(q;\lambda)\Psi$, $\Psi_s = V(q;\lambda)\Psi$ *into a new system*

$$\Psi[1]_y = U(q;\lambda)\Psi[1], \quad \Psi[1]_s = V(q;\lambda)\Psi[1].$$

The Bäcklund transformations between $(q[1], \rho[1])$ *and* (q, ρ) *are given through*

$$\begin{aligned} \rho[1] =& \rho - 2\ln_{ys}\left(\frac{\langle y_1|y_1\rangle}{\lambda_1^* - \lambda_1}\right), \\ q[1] =& q + \frac{(\lambda_1^* - \lambda_1)\psi_1^*\phi_1}{\langle y_1|y_1\rangle}. \end{aligned}$$

Single breather solution

We start with a seed solution

$$\rho[0] = -\frac{\gamma}{2}, \; q[0] = \frac{\beta}{2} \mathrm{e}^{\mathrm{i}\theta}, \; \theta = y + \frac{\gamma}{2} s.$$

Then we can get the single breather solution

$$q[1] = \frac{\beta}{2} \left[\frac{\cosh 2(\theta_{1,R} - \mathrm{i}\varphi_{1,I}) \cosh(\varphi_{1,R}) + \sin 2(\theta_{1,I} + \mathrm{i}\varphi_{1,R}) \sin(\varphi_{1,I})}{\cosh(2\theta_{1,R}) \cosh(\varphi_{1,R}) - \sin(2\theta_{1,I}) \sin(\varphi_{1,I})} \right]$$

$$x = -\frac{\gamma}{2} y - \frac{\beta^2}{8} s - 2 \ln_s \left[\cosh(2\theta_{1,R}) \cosh(\varphi_{1,R}) - \sin(2\theta_{1,I}) \sin(\varphi_{1,I}) \right],$$

$$t = -s,$$

Generally, N-breather solution:

$$q[N] = \frac{\beta}{2}\left[\frac{\det(G)}{\det(M)}\right] e^{i\theta},$$

$$x = -\frac{\gamma}{2}y - \frac{\beta^2}{8}s - 2\ln_s(\det(M)),\ t = -s,$$

where

$$M = \left(\left[\frac{e^{2(\theta_i^*+\theta_j)}}{\xi_i^* - \xi_j} + \frac{e^{2\theta_i^*}}{\xi_i^* - \chi_j} + \frac{e^{2\theta_j}}{\chi_i^* - \xi_j} + \frac{1}{\chi_i^* - \chi_j}\right] e^{-(\theta_i^*+\theta_j)}\right)_{1\le i,j\le N},$$

$$G = \left(\left[\frac{\xi_i^*+\gamma}{\xi_j+\gamma}\frac{e^{2(\theta_i^*+\theta_j)}}{\xi_i^* - \xi_j} + \frac{\xi_i^*+\gamma}{\chi_j+\gamma}\frac{e^{2\theta_i^*}}{\xi_i^* - \chi_j} + \frac{\chi_i^*+\gamma}{\xi_j+\gamma}\frac{e^{2\theta_j}}{\chi_i^* - \xi_j} + \frac{\chi_i^*+\gamma}{\chi_j+\gamma}\frac{1}{\chi_i^* - \chi_j}\right] e^{-(\theta_i^*+\theta_j)}\right)_{1\le i,j\le N}.$$

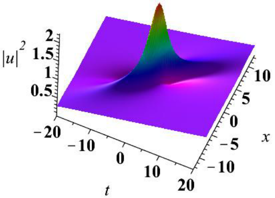
Rogue wave solution to the CSP equation

$$q[1] = \frac{\beta}{2}\left[1 + \frac{16(\mathrm{i}\beta^2 y - \beta^2 - \gamma^2)}{\beta^2\left(2y - \gamma s\right)^2 + \beta^4 s^2 + 4\gamma^2 + 4\beta^2}\right]\mathrm{e}^{\mathrm{i}\theta},$$

$$x = -\frac{\gamma}{2}y - \frac{\beta^2}{8}s - \frac{4\beta^2\left(\gamma^2 s + \beta^2 s - 2\gamma y\right)}{\beta^2\left(2y - \gamma s\right)^2 + \beta^4 s^2 + 4\gamma^2 + 4\beta^2}, \quad t = -s.$$

- $\beta^2 < \frac{\gamma^2}{3}$, then we can obtain the regular rogue wave solution
- $\beta^2 = \frac{\gamma^2}{3}$, then we can obtain the cuspon-type rogue wave
- $\beta^2 > \frac{\gamma^2}{3}$, then we can obtain the loop-type rogue wave solution

First and second-order rogue wave solutions



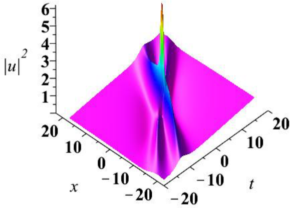


Figure: Illustration for the 1st and 2nd rogue waves of the focusing CSP equation

Bilinear equations of the defocusing complex short pulse equation

Theorem

The complex short pulse equation

$$q_{xt} + q - \frac{1}{2}\left(|q|^2 q_x\right)_x = 0$$

can be derived from bilinear equations

$$(D_s D_y - \mathrm{i}\omega D_y + \mathrm{i}\kappa D_s) g \cdot f = 0\,, \quad D_s^2 f \cdot f = \frac{1}{2}\omega^2 \left(f^2 - |g|^2\right)\,,$$

through the hodograph transformation

$$x = \omega\kappa y + \frac{\omega}{2} s - 2(\ln f)_s\,, \quad t = -s$$

and the dependent variable transformation $q = \frac{g}{f} e^{\mathrm{i}(\kappa y - \omega s)}$

Multi dark soliton to the defocusing complex short pulse equation

$$f = |A|, \quad g = \left|A^{'}\right|,$$

where the elements defined respectively by

$$a_{ij} = \delta_{ij} + \frac{1}{p_i + p_j^*} e^{\xi_i + \xi_j^*}, \quad a_{ij}^{'} = \delta_{ij} + \left(-\frac{p_i}{p_j^*}\right) \frac{1}{p_i + p_j^*} e^{\xi_i + \xi_j^*}$$

$$\xi_i = \frac{\omega}{2} p_i s + q_i \kappa y + \xi_{i0}, \ \xi_i^* = \frac{\omega}{2} p_i^* s + q_i^* \kappa y + \xi_{i0}^*$$

$$q_i = \frac{1}{p_i - \mathrm{i}}, \quad q_i^* = \frac{1}{p_i^* + \mathrm{i}}$$

where $|p_i| = 1 = e^{\mathrm{i}\phi}$, $p_i^* = e^{-\mathrm{i}\phi}$.

Define the following tau-functions for the single KP hierarchy with negative flow

$$\tau_{nk} = \left| m_{ij}^{nk} \right|_{1 \le i,j \le N} = \left| \delta_{ij} + \frac{1}{p_i + \bar{p}_j} \varphi_i^{nk} \psi_j^{nk} \right|$$

where

$$\varphi_i^{nk} = p_i^n (p_i - a)^k e^{\xi_i}$$

$$\psi_j^{nk} = \left(-\frac{1}{\bar{p}_j}\right)^n \left(-\frac{1}{\bar{p}_j + a}\right)^k e^{\bar{\xi}_j}$$

with

$$\xi_i = \frac{1}{p_i} x_{-1} + p_i x_1 + \frac{1}{p_i - a} t_a + \xi_{i0}$$

$$\bar{\xi}_j = \frac{1}{\bar{p}_j} x_{-1} + \bar{p}_j x_1 + \frac{1}{\bar{p}_j + a} t_a + \bar{\xi}_{j0}.$$

Reduction from the KP hierarchy

Then the following bilinear equations hold

$$(\frac{1}{2}D_{x_1}D_{x_{-1}} - 1)\tau_{nk}\cdot\tau_{nk} = -\tau_{n+1,k}\tau_{n-1,k}$$

$$(aD_{t_a} - 1)\tau_{n+1,k}\cdot\tau_{nk} = -\tau_{n+1,k-1}\tau_{n,k+1}$$

$$(D_{x_1}(aD_{t_a} - 1) - 2a)\tau_{n+1,k}\cdot\tau_{nk} = (D_{x_1} - 2a)\tau_{n+1,k-1}\cdot\tau_{n,k+1}$$

Objective bilinear equations:

$$(D_sD_y - \mathrm{i}\omega D_y + \mathrm{i}\kappa D_s)g\cdot f = 0\,, \quad D_s^2 f\cdot f = \frac{1}{2}\omega^2\left(f^2 - |g|^2\right)\,,$$

Reductions to the dCSP equation

By taking

$$\bar{p}_j = \frac{1}{p_j}, a = \mathrm{i}$$

we have

$$p_i + \bar{p}_i = \frac{1}{p_i} + \frac{1}{\bar{p}_i}$$

$$-\frac{\bar{p}_i}{p_i}\left(-\frac{p_i - a}{\bar{p}_i + a}\right)^2 = 1$$

thus τ_{nk} satisfies the reduction conditions

$$\partial_{x_1}\tau_{nk} = \partial_{x_{-1}}\tau_{nk}$$

$$\tau_{n-1,k+2} = \tau_{nk}.$$

Then the first bilinear equation becomes

$$\left(\frac{1}{2}D_{x_1}^2 - 1\right)\tau_{nk} \cdot \tau_{nk} = -\tau_{n+1,k}\tau_{n-1,k}$$

Reductions to the dCSP equation

Moeover, from the other bilinear equations and the above reductions, we have

$$
\begin{aligned}
& (D_{x_1}(aD_{t_a} - 1) - 2a)\tau_{n+1,k} \cdot \tau_{nk} \\
= \ & (D_{x_1} - 2a)\tau_{n+1,k-1} \cdot \tau_{n,k+1} (= \tau_{n+1,k-1}) \\
= \ & -2a\tau_{n+1,k-1} \cdot \tau_{n+1,k-1} (= \tau_{n,k+1}) \\
= \ & 2a(aD_{t_a} - 1)\tau_{n+1,k} \cdot \tau_{nk}
\end{aligned}
$$

i.e.,

$$
(D_{x_1}(D_{t_a} + \mathrm{i}) - 2\mathrm{i}D_{t_a})\tau_{n+1,k} \cdot \tau_{nk} = 0
$$

By taking $|p_i| = 1, \bar{\xi}_{j0} = \xi^*_{j0}$,where * means complex conjugate, we have

$$\tau^*_{n0} = \tau_{-n,0}$$

$$\tau_{n0} = \left| \delta_{ij} + \frac{1}{p_i + p^*_j} (-\frac{p_i}{p^*_j})^n e^{\xi_i + \xi^*_j} \right|_{1 \leq i,j \leq N}$$

By defining

$$f = \tau_{00}, g = \tau_{10}$$

we get

$$(\frac{1}{2} D^2_{x_1} - 1) f \cdot f = -gg^*$$

$$(D_{x_1} D_{t_a} + \mathrm{i} D_{x_1} - 2\mathrm{i} D_{t_a}) g \cdot f = 0.$$

Finally, by setting $t_a = \kappa y$, $2x_1 = \omega s$,the above bilinear equations are converted into

$$(D^2_s - \frac{\omega^2}{2}) f \cdot f = -\frac{\omega^2}{2} gg^*$$

Summary for the focusing and defocusing CSP equation

- The bright soliton solution to the focusing CSP equation can be obtained from the reduction of the two-component KP hierarchy or from the Darboux transformation
- The rogue wave solution to the focusing CSP equation can be obtained from the Darboux transformation, we are working on the higher order rogue wave solutions by Hirota's bilinear method
- The dark soliton solution to the defocusing CSP equation can be obtained from the reduction of the one-component KP hierarchy or from the Darboux transformation

Integrable semi-discrete complex short pulse equation

Theorem

Bilinear equations

$$\left\{\begin{array}{l} \frac{1}{a}D_s(g_{k+1}\cdot f_k - g_k\cdot f_{k+1}) = g_{k+1}f_k + g_k f_{k+1}\,, \\ D_s^2 f_k\cdot f_k = \frac{1}{2}g_k g_k^*\,. \end{array}\right.$$

give semi-discrete complex SP equation

$$\left\{\begin{array}{l} \frac{d}{dt}(q_{k+1}-q_k) = \frac{1}{2}(x_{k+1}-x_k)(q_{k+1}+q_k)\,, \\ \frac{d}{dt}(x_{k+1}-x_k) = -\frac{1}{2}(|q_{k+1}|^2 - |q_k|^2)\,. \end{array}\right.$$

through transformations

$$q_k = \frac{g_k}{f_k}, \quad x_k = 2ka - 2(\ln f_k)_s.$$

Multi-soliton solution:

$$f_k = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \quad g_k = \begin{vmatrix} A & I & \Phi^T \\ -I & B & 0^T \\ 0 & C_1 & 0 \end{vmatrix},$$

where the elements defined respectively by

$$a_{ij} = \frac{1}{2(p_i^{-1} + p_j^{*-1})} e^{\xi_i + \bar{\xi}_j}, \quad b_{ij} = \frac{\alpha_i^* \alpha_j}{2(p_j^{-1} + p_i^{*-1})}$$

$$e^{\xi_i} = \left(\frac{1 + ap_i}{1 - ap_i}\right)^k \exp(\frac{1}{p_i} s + \xi_{i0}), \; e^{\xi_j^*} = \left(\frac{1 + ap_j^*}{1 - ap_j^*}\right)^k \exp(\frac{1}{p_i^*} s + \bar{\xi}_{j0}).$$

Lax pair to the semi-discrete CSP equation

$$\Psi_{k+1} = U_k \Psi_k, \quad \Psi_{k,t} = V_k \Psi_k\,,$$

with

$$U_k = \begin{pmatrix} 1 - \mathrm{i}\lambda\delta_k & -\mathrm{i}\lambda(q_{k+1} - q_k) \\ -\mathrm{i}\lambda(q^*_{k+1} - q^*_k) & 1 + \mathrm{i}\lambda\delta_k \end{pmatrix}$$

$$V_k = \begin{pmatrix} \frac{\mathrm{i}}{4\lambda} & -\frac{1}{2}q_k \\ \frac{1}{2}q^*_k & -\frac{\mathrm{i}}{4\lambda} \end{pmatrix}$$

- The compatibility condition $dU_k/dt + U_k V_k - V_{k+1} U_k = 0$ gives the semi-discrete CSP equation

Fully discrete complex short pulse equation

Bilinear equations

$$\begin{cases} g_{k+1}^{l+1} f_k^l - g_k^{l+1} f_{k+1}^l - g_{k+1}^l f_k^{l+1} + g_k^l f_{k+1}^{l+1} \\ = ab(g_{k+1}^{l+1} f_k^l + g_k^{l+1} f_{k+1}^l + g_{k+1}^l f_k^{l+1} + g_k^l f_{k+1}^{l+1}) \\ f_k^{l+1} f_k^{l-1} - f_k^l f_k^l = b^2 g_k^l \bar{g}_k^l \end{cases}$$

give the fully discrete complex SP equation

$$\begin{cases} (1-ab)(q_k^l + q_{k+1}^{l+1}) = (1+ab)\left(q_k^{l+1} + q_{k+1}^l\right)(1+(\delta_k^l - 2a)b) \\ \dfrac{1+(\delta_k^l - 2a)b}{1+(\delta_k^{l-1} - 2a)b} = \dfrac{1+b^2 q_k^l \bar{q}_k^l}{1+b^2 q_{k+1}^l \bar{q}_{k+1}^l} \end{cases}$$

through transformations

$$q_k^l = \frac{g_k^l}{f_k^l}, \quad \delta_k^l = 2a + \frac{1}{b}\left(\frac{f_k^{l+1} f_{k+1}^l}{f_{k+1}^{l+1} f_k^l} - 1\right).$$

Conclusion and further topics

- We have proposed a focusing and defocusing complex short pulse equation to describe the propagation of ultra-short pulse in optical fibers
- The multi-bright and multi-dark soliton solutions are obtained from the reductions of the KP hierarchies
- The soliton, breather and rogue wave solutions are constructed via the Darboux transformation

Conclusion and further topics

- We have proposed a focusing and defocusing complex short pulse equation to describe the propagation of ultra-short pulse in optical fibers
- The multi-bright and multi-dark soliton solutions are obtained from the reductions of the KP hierarchies
- The soliton, breather and rogue wave solutions are constructed via the Darboux transformation
- Further topic 1: Physical applications
- Further topic 2: Self-adaptive moving method based on integrable discretizations
- Further topic 3: Studies for the coupled CSP equation